VERTEX REPRESENTATIONS VIA FINITE GROUPS AND THE MCKAY CORRESPONDENCE

IGOR B. FRENKEL, NAIHUAN JING, AND WEIQIANG WANG

1. INTRODUCTION

It was first realized in [FK] and [S1] that the basic representation V of an affine Lie algebra $\widehat{\mathfrak{g}}$ of ADE type can be constructed from the root lattice Q of the corresponding finite dimensional Lie algebra $\mathfrak{g}$ as follows:

$$(1.1) \qquad V = S(Q \bigotimes \mathbb{C}[t^{-1}]t^{-1}) \bigotimes \mathbb{C}[Q],$$

where the first factor is a symmetric algebra and the second one is a group algebra. The affine algebra $\widehat{\mathfrak{g}}$ contains a Heisenberg algebra $\widehat{\mathfrak{h}}$. One can define the so-called vertex operators $X(\alpha, z)$ associated to $\alpha \in Q$ acting on V essentially using the Heisenberg algebra $\widehat{\mathfrak{h}}$. The representation of $\widehat{\mathfrak{g}}$ on V is then obtained from the action of the Heisenberg algebra $\widehat{\mathfrak{h}}$ and the vertex operators $X(\alpha, z)$ associated to α in the root system of $\mathfrak{g}$.

This construction was extended in [F2] to more general lattices to provide vertex representations of affinization of Kac-Moody Lie algebras. The special case of affinization of affine Lie algebras called toroidal Lie algebras was discussed further in [MRY].

Another important special case of vertex representations is given by the standard lattice $\mathbb{Z}^N$. The vertex operators corresponding to the unit vectors in $\mathbb{Z}^N$ give rise to a representation of an infinite-dimensional Clifford algebra [F1], the relation known as boson-fermion correspondence. In the special case $N = 1$, the transition matrix between the monomial bases for representations of Heisenberg and Clifford algebras yields the character tables of symmetric groups S_n for all n [F3] (see [J1]).

Vertex operators have appeared and played an important role in many diversed fields of mathematics and physics (cf. [FLM] and references therein). Surprisingly, they also appeared in recent works of Nakajima (cf. [N] and references therein) and independently of Grojnowski [Gr] on the (homology groups of) Hilbert schemes of points on a surface X. In particular when X is a minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$ of the simple quotient singularity of $\mathbb{C}^2$ by a finite subgroup Γ of SU_2, they were able to realize geometrically the vertex representations of affine Lie algebras.

Vertex operators showed up in equivariant K-theory as well. Partly motivated by a footnote in [Gr], Segal [S2] outlined the S_n-equivariant K-theory of the n-th direct product X^n of X. Generalizing [S2], the third-named author [W] studied in detail the Γ_n-equivariant K-theory of X^n for a Γ-space X, where Γ_n is the so-called wreath product which is the semi-direct product of the symmetric group S_n and the n-th direct product of a finite group Γ. Among other results a construction of vertex representations was proposed in [W] in terms of representations of wreath products; in particular when Γ is a subgroup of SU_2, a realization of affine algebras thus obtained can be regarded as a new form of McKay correspondence [Mc].

The goal of this paper is to establish firmly such a finite group theoretic approach of the vertex representations corresponding to a very general class of lattices, including the (affine) root lattices of ADE type and the standard lattice $\mathbb{Z}^N$ as special cases. It was conjectured [W] that there exists a natural isomorphism from the representation ring of Γ_n for a finite subgroup Γ of SU_2 to the homology group of the Hilbert scheme of n points on $\widehat{\mathbb{C}^2/\Gamma}$ after the dimensions of these two spaces were shown to be the same. In this way, the group theoretic realization of vertex representations for affine algebras in this paper might be regarded as a counterpart of the geometric construction given in [Gr, N] for the surface $\widehat{\mathbb{C}^2/\Gamma}$.

Let us explain the contents of this paper in more detail. Let Γ be an arbitrary finite group. Denote by $R_{\mathbb{Z}}(\Gamma_n)$ the representation ring of the wreath product Γ_n and let $R(\Gamma_n) = R_{\mathbb{Z}}(\Gamma_n) \otimes_{\mathbb{Z}} \mathbb{C}$. We set

$$R_{\Gamma} = \bigoplus_{n \geq 0} R(\Gamma_n).$$

The representation theory of wreath products was developed by Specht (cf. [M2, Z]). Given a self-dual virtual character ξ in $R(\Gamma)$, we introduce a *weighted* bilinear form on $R(\Gamma_n)$ and an induced one on R_{Γ}

denoted by $\langle\ ,\ \rangle_\xi$. When ξ is trivial, it reduces to the standard bilinear form on the representation ring of a finite group.

It turns out that the spaces constructed from the representation theory of Γ_n reproduce those appearing in vertex representations (see Eqn. (1.1)). In fact one can construct the symmetric algebra S_Γ associated to the lattice $R_{\mathbb{Z}}(\Gamma)$. We define a natural bilinear form $\langle\ ,\ \rangle'_\xi$ induced from the ξ-weighted bilinear form on R_Γ. We define a characteristic map ch from $(R_\Gamma, \langle\ ,\ \rangle_\xi)$ to $(S_\Gamma, \langle\ ,\ \rangle'_\xi)$ and show that it is an isometry. The map ch for ξ being a trivial character was considered in [M1, M2].

We introduce an infinite-dimensional Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ acting on S_Γ associated to Γ and ξ, and thus regard S_Γ as the Fock space of $\widehat{\mathfrak{h}}_{\Gamma,\xi}$. We give a group theoretic realization [W] of this Heisenberg algebra acting on R_Γ which is compatible with the one on S_Γ via the characteristic map ch.

Denote by $\mathcal{F}_\Gamma$ the tensor product of R_Γ and the group algebra of $R_{\mathbb{Z}}(\Gamma)$ which is the full space of a vertex representation. Given $\gamma \in R_{\mathbb{Z}}(\Gamma)$, we define a generating function $X(\gamma, z)$ of group theoretic operators acting on R_Γ. We prove that $X(\gamma, z)$ thus defined is precisely the vertex operator associated to γ which can be essentially constructed in terms of the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$. We then calculate the product of two vertex operators of the form $X(\gamma, z), \gamma \in R_{\mathbb{Z}}(\Gamma)$, which are equivalent to the commutation relations among the components of $X(\gamma, z)$.

Now we specialize to the important case when Γ is a finite group of SU_2 and ξ is twice the trivial character minus the character of a two-dimensional natural representation of Γ. According to McKay [Mc], the lattice $(R_{\mathbb{Z}}(\Gamma), \langle\ ,\ \rangle_\xi)$ is positive semi-definite and its radical is one-dimensional with a generator δ given by the regular character of Γ. Furthermore $(R_{\mathbb{Z}}(\Gamma), \langle\ ,\ \rangle_\xi)$ can be identified with the root lattice of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ of ADE type. Our general construction when specialized to this case gives a group theoretic realization of the toroidal Lie algebra associated to $\mathfrak{g}$.

The sublattice $\overline{R}_{\mathbb{Z}}(\Gamma)$ of $R_{\mathbb{Z}}(\Gamma)$ generated by non-trivial irreducible characters of Γ can be identified with the root lattice of $\mathfrak{g}$. It follows that the lattice $R_{\mathbb{Z}}(\Gamma)$ is a direct sum of $\overline{R}_{\mathbb{Z}}(\Gamma)$ and the lattice $\mathbb{Z}\delta$ with zero bilinear form generated by δ. Therefore $\mathcal{F}_\Gamma$ can be decomposed as a tensor product of the space $\overline{\mathcal{F}}_\Gamma$ associated to the lattice $\overline{R}_{\mathbb{Z}}(\Gamma)$ with the space associated to the lattice $\mathbb{Z}\delta$. This gives a new form of "McKay correspondence": starting with a finite subgroup $\Gamma \subset SU_2$ we

have been able to construct the basic representation on $\overline{\mathcal{F}}_\Gamma$ of the affine Lie algebra $\widehat{\mathfrak{g}}$ with the Dynkin diagram corresponding to Γ.

If we specialize ξ to be the trivial character of Γ, then the lattice $(R_{\mathbb{Z}}(\Gamma), \langle \ , \ \rangle_\xi)$ is just the standard lattice $\mathbb{Z}^N$ where N is the number of irreducible characters of Γ. The vertex operators corresponding to irreducible characters of Γ generate an infinite-dimensional Clifford algebra. In this case the transition matrix between the monomial bases of Heisenberg and Clifford algebras yields the character table of the wreath product Γ_n for all n, generalizing the symmetric group picture.

We announce some further directions we are currently pursuing. One is on the q-deformation of our construction while another is on the generalized McKay correspondence and affine Lie algebras of non-simply-laced type. We will also generalize the results in [J1, J2] to the wreath product setting. In the end we pose the very important problem to give a group theoretic construction of the whole vertex algebra structure on $\mathcal{F}_\Gamma$ [B, FLM].

The paper is organized as follows. In Sect. 2 we review the theory of wreath products. In Sect. 3 we introduce the weighted bilinear form on R_Γ. In Sect. 4 we define the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ and its Fock space together with some natural basis. In Sect. 5 we establish the isometry between R_Γ and S_Γ. In Sect. 6 we define the vertex operators acting on R_Γ. In Sect. 7 we calculate products of vertex operators and establish a new McKay correspondence. In Sect. 8 we derive the character tables of Γ_n from the vertex operator approach.

2. Preliminaries on wreath products

2.1. **The wreath product Γ_n.** Let Γ be a finite group with $r + 1$ conjugacy classes. We denote by Γ^* the set of complex irreducible characters and by Γ_* the set of conjugacy classes. The character value $\gamma(c)$ of $\gamma \in \Gamma^*$ at a conjugacy class $c \in \Gamma_*$ yields the character table $\{\gamma(c)\}$ of Γ.

The space of class functions on Γ is given by

$$R(\Gamma) = \bigoplus_{i=0}^{r} \mathbb{C}\gamma_i,$$

where γ_i are all the inequivalent irreducible characters of Γ and in particular γ_0 is the trivial character. We denote by $R_{\mathbb{Z}}(\Gamma)$ the integral combination of irreducible characters of Γ. Denote by $c^i, i = 0, \ldots, r$ the distinct conjugacy classes in Γ_*, in particular c^0 is the identity conjugacy class.

For $c \in \Gamma_*$ let ζ_c be the order of the centralizer of an element in the class c, so the order of the class is then $|c| = |\Gamma|/\zeta_c$. The usual bilinear form on $R(\Gamma)$ is defined as follows:

$$\langle f, g \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x)g(x^{-1}) = \sum_{c \in \Gamma_*} \zeta_c^{-1} f(c)g(c^{-1}),$$

where c^{-1} denotes the conjugacy class $\{x^{-1}, x \in c\}$. Clearly $\zeta_c = \zeta_{c^{-1}}$. We will refer to this bilinear form as the *standard* one on $R(\Gamma)$. We will often write $\langle \ , \ \rangle$ for $\langle \ , \ \rangle_\Gamma$ when no ambiguity may arise. It is well known that

$$\langle \gamma_i, \gamma_j \rangle = \delta_{ij},$$

(2.1)
$$\sum_{\gamma \in \Gamma^*} \gamma(c')\gamma(c^{-1}) = \delta_{c,c'}\zeta_c, \quad c, c' \in \Gamma_*.$$

Thus $R_{\mathbb{Z}}(\Gamma)$ endowed with this bilinear form becomes an integral lattice in $R(\Gamma)$.

Remark 2.1. Besides the standard bilinear form on a finite group one can introduce a standard Hermitian form which is also useful in our constructions. Throughout the paper we have chosen to use bilinear forms instead of Hermitian forms.

Given a positive integer n, let $\Gamma^n = \Gamma \times \cdots \times \Gamma$ be the n-th direct product of Γ, and let Γ^0 be the trivial group. The symmetric group S_n acts on Γ^n by permutations: $\sigma(g_1, \cdots, g_n) = (g_{\sigma^{-1}(1)}, \cdots, g_{\sigma^{-1}(n)})$. The wreath product of Γ with S_n is defined to be the semi-direct product

$$\Gamma_n = \{(g, \sigma) | g = (g_1, \cdots, g_n) \in \Gamma^n, \sigma \in S_n\}$$

with the multiplication

$$(g, \sigma) \cdot (h, \tau) = (g \, \sigma(h), \sigma\tau).$$

Note that $\Gamma_n = S_n$ when $\Gamma = 1$, and Γ_n is the hyperoctahedral group of rank n when $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

2.2. **Conjugacy classes of Γ_n.** Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ be a partition of integer $|\lambda| = \lambda_1 + \cdots + \lambda_l$, where $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$. The integer l is called the *length* of the partition λ and is denoted by $l(\lambda)$. We will identify the partition $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ with $(\lambda_1, \lambda_2, \cdots, \lambda_l, 0, \cdots, 0)$. We will also make use of another notation for partitions:

$$\lambda = (1^{m_1} 2^{m_2} \cdots),$$

where m_i is the number of parts in λ equal to i.

We will use partitions indexed by Γ_* and Γ^*. For a finite set X and $\rho = (\rho(x))_{x \in X}$ a family of partitions indexed by X, we write

$$\|\rho\| = \sum_{x \in X} |\rho(x)|.$$

Sometimes it is convenient to regard $\rho = (\rho(x))_{x \in X}$ as a partition-valued function on X. We denote by $\mathcal{P}(X)$ the set of all partitions indexed by X and by $\mathcal{P}_n(X)$ the set of all partitions in $\mathcal{P}(X)$ such that $\|\rho\| = n$.

The conjugacy classes of Γ_n can be described in the following way. Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \cdots, g_n) \in \Gamma^n$, $\sigma \in S_n$. The permutation σ is written as a product of disjoint cycles. For each such cycle $y = (i_1 i_2 \cdots i_k)$ the element $g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$ is determined up to conjugacy in Γ by g and y, and will be called the *cycle-product* of x corresponding to the cycle y. For any conjugacy class c and each integer $i \geq 1$, the number of i-cycles in σ whose cycle-product lies in c will be denoted by $m_i(c)$. Denote by $\rho(c)$ the partition $(1^{m_1(c)} 2^{m_2(c)} \ldots)$, $c \in \Gamma_*$. Then each element $x = (g, \sigma) \in \Gamma_n$ gives rise to a partition-valued function $(\rho(c))_{c \in \Gamma_*} \in \mathcal{P}(\Gamma_*)$ such that $\sum_{i,c} i m_i(c) = n$. The partition-valued function $\rho = (\rho(c))_{c \in G_*}$ is called the *type* of x. It is known (cf. [M2]) that any two elements of Γ_n are conjugate in Γ_n if and only if they have the same type.

Given a partition $\lambda = (1^{m_1} 2^{m_2} \ldots)$, we define

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

We note that z_λ is the order of the centralizer of an element of cycle-type λ in $S_{|\lambda|}$. The order of the centralizer of an element $x = (g, \sigma) \in \Gamma_n$ of the type $\rho = (\rho(c))_{c \in \Gamma_*}$ is

$$Z_\rho = \prod_{c \in \Gamma_*} z_{\rho(c)} \zeta_c^{l(\rho(c))}.$$

2.3. **Hopf algebra structure on R_Γ.** We define

$$R_\Gamma = \bigoplus_{n \geq 0} R(\Gamma_n).$$

R_Γ carries a natural Hopf algebra structure (cf. e.g. [M1, Z, W]) with multiplication defined by the composition

$$m : R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{\cong} R(\Gamma_n \times \Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m}),$$

and comultiplication defined by the composition

$$\Delta : R(\Gamma_n) \xrightarrow{Res} \bigoplus_{m=0}^{n} R(\Gamma_{n-m} \times \Gamma_m) \xrightarrow{\cong} \bigoplus_{m=0}^{n} R(\Gamma_{n-m}) \bigotimes R(\Gamma_m).$$

Here $Ind : R(\Gamma_n \times \Gamma_m) \longrightarrow R(\Gamma_{n+m})$ is the induction functor and $Res : R(\Gamma_n) \longrightarrow R(\Gamma_{n-m} \times \Gamma_m)$ is the restriction functor.

The standard bilinear form in R_Γ is defined in terms of those on $R(\Gamma_n)$ as follows:

$$\langle u, v \rangle = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\Gamma_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in \Gamma_n$.

3. WEIGHTED BILINEAR FORMS ON $R(\Gamma)$ AND $R(\Gamma_n)$

3.1. **A weighted bilinear form on $R(\Gamma)$.** Let us fix a class function $\xi \in R(\Gamma)$. The multiplication in $R(\Gamma)$ corresponding to the tensor product of two representations will be denoted by $*$.

We denote by $a_{ij} \in \mathbb{C}$ the (virtual) multiplicities of γ_j in $\xi * \gamma_i$. In other words, we have the following decomposition

$$(3.1) \qquad \xi * \gamma_i = \sum_{j=0}^{r} a_{ij} \gamma_j.$$

We denote by A the $(r+1) \times (r+1)$ matrix $(a_{ij})_{0 \leq i,j \leq r}$.

We introduce the following weighted bilinear form

$$\langle f, g \rangle_\xi = \langle \xi * f, g \rangle_\Gamma, \quad f, g \in R(\Gamma).$$

We also have an alternative formula:

$$\langle f, g \rangle_\xi = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} \xi(x) f(x) g(x^{-1})$$

$$(3.2) \qquad\qquad = \sum_{c \in \Gamma_*} \zeta_c^{-1} \xi(c) f(c) g(c^{-1}).$$

In other words, $\langle \ , \ \rangle_\Gamma$ is the average of the character $\xi * f * \overline{g}$.

In particular, Eqn. (3.1) can be reformulated as

$$\langle \gamma_i, \gamma_j \rangle_\xi = a_{ij}.$$

Throughout this paper we will always assume that ξ is a *self-dual*, i.e. $\overline{\xi} = \xi$, or equivalently $\xi(x) = \xi(x^{-1}), x \in \Gamma$. The self-duality of ξ together with (3.2) implies that

$$a_{ij} = a_{ji},$$

i.e. A is a symmetric matrix. An equivalent formula for $\langle \ , \ \rangle_\xi$ is

$$(3.3) \qquad \langle f, g \rangle_\xi \;=\; \sum_{c \in \Gamma_*} \zeta_c^{-1} \xi(c) f(c^{-1}) g(c).$$

Remark 3.1. If ξ is the trivial character γ_0, then the weighted bilinear form becomes the standard one on $R(\Gamma)$.

3.2. **A weighted bilinear form on $R(\Gamma_n)$.** Given a representation V of Γ with character $\gamma \in R(\Gamma)$, the n-th outer tensor product $V^{\otimes n}$ of V can be regarded naturally as a representation of the wreath product Γ_n whose character will be denoted by $\eta_n(\gamma)$: the direct product Γ^n acts on $\gamma^{\otimes n}$ factor by factor while S_n by permuting the n factors. Denote by ε_n the (1-dimensional) sign representation of Γ_n on which Γ^n acts trivially while S_n acts as sign representation. We denote by $\varepsilon_n(\gamma) \in R(\Gamma_n)$ the character of the tensor product of ε_n and $V^{\otimes n}$.

We may extend naturally η_n to a map from $R(\Gamma)$ to $R(\Gamma_n)$ (cf. [W]). In particular, if β and γ are characters of representations V and W of Γ respectively, then

$$(3.4) \quad \eta_n(\beta - \gamma) = \sum_{m=0}^{n} (-1)^m Ind_{\Gamma_{n-m} \times \Gamma_m}^{\Gamma_n} [\eta_{n-m}(\beta) \otimes \varepsilon_m(\gamma)].$$

We introduce a *weighted bilinear form* on $R(\Gamma_n)$ by letting

$$\langle f, g \rangle_{\xi, \Gamma_n} = \langle \eta_n(\xi) * f, g \rangle_{\Gamma_n}, \quad f, g \in R(\Gamma_n).$$

We shall see in Corollary 5.3 that $\eta_n(\xi)$ is self-dual. It follows that the bilinear form $\langle \ , \ \rangle_\xi$ is symmetric.

Remark 3.2. When $n = 1$, this weighted bilinear form obviously reduces to the weighted bilinear form defined on $R(\Gamma)$.

On $R_\Gamma = \bigoplus_n R(\Gamma_n)$ a symmetric bilinear form is given by

$$\langle u, v \rangle_\xi = \sum_{n \geq 0} \langle u_n, v_n \rangle_{\xi, \Gamma_n},$$

where $u = \sum_n u_n$ and $v = \sum_n v_n$ with $u_n, v_n \in \Gamma_n$.

3.3. **A McKay specialization.** Denote by $d_i = \gamma_i(c^0)$ the dimension of the irreducible representation of Γ corresponding to the character γ_i. The following proposition appeared in [St] which was motivated by a special important case observed first by McKay [Mc].

Proposition 3.1. *The column vector*

$$v_i = (\gamma_0(c^i), \gamma_1(c^i), \ldots, \gamma_r(c^i))^t \quad (i = 0, \ldots, r)$$

is an eigenvector of the matrix A with eigenvalue $\xi(c^i)$. In particular $(d_0, d_1, \ldots, d_r)^t$ is an eigenvector of A with eigenvalue $\xi(c^0)$.

Denote by E the character matrix $[v_0, v_1, \dots, v_r]$ and D the diagonal matrix $\mathrm{diag}(\xi(c^0), \dots, \xi(c^r))$. The proposition above can be reformulated as

$$AE = ED.$$

We fix an irreducible faithful complex representation π of Γ of dimension d. We still assume that π is self-dual. We set

$$\xi = d\gamma_0 - \pi.$$

In this case, the weighted bilinear form $\langle \ , \ \rangle_\xi$ on R_Γ become semi-positive definite. The radical of this bilinear form is one-dimensional and spanned by the character of the regular representation of Γ

$$\delta = \sum_{i=0}^{r} d_i \gamma_i.$$

We further specialize to the case in which π embeds Γ into SU_2, i.e. $d = 2$. The classification of finite subgroups of SU_2 is well known. The following is a complete list of them: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. We recall the following well-known facts [Mc]: $a_{ii} = 2$ for all i; if $\Gamma \neq \mathbb{Z}/2\mathbb{Z}$ and $i \neq j$ then $a_{ij} = 0$ or -1. If $\Gamma \equiv \mathbb{Z}/2\mathbb{Z}$ then $a_{01} = -2$.

We associate a diagram with vertices corresponding to elements γ_i in Γ^*. We draw one edge (resp. two edges) between the i-th and j-th vertices if $a_{ij} = -1$ (resp. -2). According to McKay [Mc], the associated diagram can be identified with affine Dynkin diagram of ADE type and the matrix A is the corresponding affine Cartan matrix.

4. Heisenberg algebras and Γ_n

4.1. **Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$.** Let $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ be the infinite dimensional Heisenberg algebra over $\mathbb{C}$, associated with Γ and ξ, with generators $a_m(\gamma), m \in \mathbb{Z}, \gamma \in \Gamma^*$ and a central element C subject to the following commutation relations:

(4.1) $[a_m(\gamma), a_n(\gamma')] = m\delta_{m,-n}\langle \gamma, \gamma' \rangle_\xi C, \quad m, n \in \mathbb{Z}, \ \gamma, \gamma' \in \Gamma^*.$

It is convenient to extend $a_m(\gamma)$ to all $\gamma = \sum_{i=0}^{r} s_i \gamma_i \in R(\Gamma)$ $(s_i \in \mathbb{C})$ by linearity: $a_m(\gamma) = \sum_i s_i \, a_m(\gamma_i)$.

Denote by R_0 the radical of the bilinear form $\langle \cdot, \cdot \rangle_\xi$ in $R(\Gamma)$. We note that the Heisenberg algebra may contain a large center since the bilinear form $\langle \cdot, \cdot \rangle_\xi$ may degenerate. In this sense we have abused the notion of Heisenberg algebra. The center of $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ is spanned by C together with $a_m(\gamma), \gamma \in R_0, m \in \mathbb{Z}$.

For $m \in \mathbb{Z}, c \in \Gamma_*$ we define

$$a_m(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1}) a_m(\gamma).$$

From the orthogonality of the irreducible characters of Γ (2.1) it follows that

$$a_m(\gamma) = \sum_{c \in \Gamma_*} \zeta_c^{-1} \gamma(c) a_m(c).$$

Thus $a_n(c)$ $(n \in \mathbb{Z}, c \in \Gamma_*)$ and C form a new basis for the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$.

Proposition 4.1. *The commutation relations among the new basis for $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ are given by*

$$[a_m(c'^{-1}), a_n(c)] \;=\; m \delta_{m,-n} \delta_{c',c} \zeta_c \xi(c) C, \quad c, c' \in \Gamma_*.$$

Proof. We calculate by using Eqns. (4.1), (3.3) and (2.1) that

$$
\begin{aligned}
&[a_m(c'^{-1}), a_n(c)] \\
&= \sum_{\gamma',\gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})[a_m(\gamma'), a_n(\gamma)] \\
&= m\delta_{m,-n} \sum_{\gamma',\gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})\langle \gamma', \gamma \rangle_\xi C \\
&= m\delta_{m,-n} \sum_{\gamma',\gamma \in \Gamma^*} \gamma'(c')\gamma(c^{-1})\left(\sum_{\nu \in \Gamma_*} \zeta_\nu^{-1} \xi(\nu)\gamma(\nu)\gamma'(\nu^{-1}) \right) C \\
&= m\delta_{m,-n} \sum_{\nu \in \Gamma_*} \zeta_\nu^{-1}\xi(\nu)\left(\sum_{\gamma' \in \Gamma^*} \gamma'(c')\gamma'(\nu^{-1}) \right)\left(\sum_{\gamma \in \Gamma^*} \gamma(c^{-1})\gamma(\nu) \right) C \\
&= m\delta_{m,-n} \sum_{\nu \in \Gamma_*} \zeta_\nu^{-1}\xi(\nu)\delta_{c',\nu}\zeta_\nu \delta_{c,\nu}\zeta_\nu C \\
&= m\delta_{m,-n}\delta_{c,c'}\zeta_c \xi(c) C.
\end{aligned}
$$

$\square$

4.2. **Action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ on S_Γ and $\overline{S}_\Gamma$.** Denote by S_Γ the symmetric algebra generated by $a_{-n}(\gamma), n \in \mathbb{N}, \gamma \in \Gamma^*$. There is a natural degree operator on S_Γ

$$\deg(a_{-n}(\gamma)) = n,$$

which makes S_Γ into a $\mathbb{Z}_+$-graded space.

We define an action of $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ on S_Γ as follows: $a_{-n}(\gamma), n > 0$ acts as multiplication operator on S_Γ and C as the identity operator; $a_n(\gamma)$,

$n \geq 0$ acts as a derivation of algebra

$$a_n(\gamma).a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\dots a_{-n_k}(\alpha_k)$$

$$= \sum_{i=1}^{k} \delta_{n,n_i}\langle\gamma,\alpha_i\rangle_\xi a_{-n_1}(\alpha_1)a_{-n_2}(\alpha_2)\dots \check{a}_{-n_i}(\alpha_i)\dots a_{-n_k}(\alpha_k).$$

Here $n_i > 0, \alpha_i \in R(\Gamma)$ for $i = 1, \dots, k$, and $\check{a}_{-n_i}(\alpha_i)$ means the very term is deleted. In other word, the operator $a_n(\gamma), n > 0, \gamma \in R^0$ acts as 0, and $a_n(\gamma), n > 0, \gamma \in R(\Gamma) - R^0$ acts as certain non-zero differential operator. Note that S_Γ is not an irreducible representation over $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ in general since the bilinear form $\langle\ ,\ \rangle_\xi$ may be degenerate.

Denote by S_Γ^0 the ideal in the symmetric algebra S_Γ generated by $a_{-n}(\gamma), n \in \mathbb{N}, \gamma \in R_0$. Denote by $\overline{S}_\Gamma$ the quotient S_Γ/S_Γ^0. It follows from the definition that S_Γ^0 is a subrepresentation of S_Γ over the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$. In particular, this induces a Heisenberg algebra action on $\overline{S}_\Gamma$ which is irreducible.

We denote by 1 the unit in the symmetric algebra S_Γ. By abuse of notation we will denote by 1 its image in the quotient $\overline{S}_\Gamma$. The element 1 is the highest weight vector in $\overline{S}_\Gamma$.

4.3. The bilinear form on S_Γ.

The space S_Γ admits a bilinear form $\langle\ ,\ \rangle'_\xi$ characterized by

$$(4.2) \qquad \langle 1,1\rangle'_\xi = 1, \quad a_n(\gamma)^* = a_{-n}(\gamma), \qquad n \in \mathbb{Z}\backslash 0.$$

Here $a_n(\gamma)^*$ denotes the adjoint of $a_n(\gamma)$.

For any partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\gamma \in \Gamma^*$, we define

$$a_{-\lambda}(\gamma) = a_{-\lambda_1}(\gamma)a_{-\lambda_2}(\gamma)\dots.$$

For $\rho = (\rho(\gamma))_{\gamma\in\Gamma^*} \in \mathcal{P}(\Gamma^*)$, we define

$$a_{-\rho} = \prod_{\gamma\in\Gamma^*} a_{-\rho(\gamma)}(\gamma).$$

It is clear that $a_{-\rho}, \rho \in \mathcal{P}(\Gamma^*)$ consist of a $\mathbb{C}$-basis for S_Γ.

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and $c \in \Gamma_*$, we define

$$a_{-\lambda}(c) = a_{-\lambda_1}(c)a_{-\lambda_2}(c)\dots.$$

For any $\rho = (\rho(c))_{c\in\Gamma_*} \in \mathcal{P}(\Gamma_*)$, we further define

$$a'_{-\rho} = \prod_{c\in\Gamma_*} a_{-\rho(c)}(c).$$

The elements $a'_{-\rho}, \rho \in \mathcal{P}(\Gamma_*)$ provide a new $\mathbb{C}$-basis for S_Γ.

We define $\overline{\rho} \in \mathcal{P}(\Gamma_*)$ by assigning to $c \in \Gamma_*$ the partition $\rho(c^{-1})$, which is the composition of ρ with the involution on Γ_* given by $c \mapsto c^{-1}$. It follows from Proposition 4.1 that

$$(4.3) \quad \langle a'_{-\rho'}, a'_{-\overline{\rho}} \rangle'_\xi = \delta_{\rho',\rho} Z_\rho \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}, \quad \rho', \rho \in \mathcal{P}(\Gamma_*).$$

Remark 4.1. S_Γ^0 can be characterized as the radical of the bilinear form $\langle\ ,\ \rangle'_\xi$ in S_Γ. Thus the bilinear form $\langle\ ,\ \rangle'_\xi$ descends to $\overline{S}_\Gamma$.

5. ISOMETRY BETWEEN R_Γ AND S_Γ

5.1. **The characteristic map ch.** Let $\Psi : \Gamma_n \to S_\Gamma$ be the map defined by $\Psi(x) = a'_{-\rho}$ if $x \in \Gamma_n$ is of type ρ. Given $y \in \Gamma_m$ of type ρ', we may regard $x \times y \in \Gamma_n \times \Gamma_m$ to be in Γ_{n+m} of type $\rho \cup \rho'$ so that

$$\Psi(x \times y) = \Psi(x)\Psi(y).$$

We define a $\mathbb{C}$-linear map $ch : R_\Gamma \longrightarrow S_\Gamma$ by letting

$$ch(f) = \langle f, \Psi \rangle_{\Gamma_n} = \sum_{\rho \in \mathcal{P}(\Gamma_*)} Z_\rho^{-1} f_\rho a'_{-\rho},$$

where f_ρ is the value of f at elements of type ρ. The map ch is called the *characteristic map*.

We may think of $a_{-n}(\gamma), n > 0, \gamma \in \Gamma^*$ as the n-th power sum in a sequence of variables $y_\gamma = (y_{i\gamma})_{i \geq 1}$. In this way we identify the space S_Γ with the space Λ_Γ of symmetric functions indexed by Γ^* (cf. [M2]). In particular given a partition λ we denote by $s_\lambda(\gamma)$ the Schur function associated to y_γ. By abuse of notation, we denote by $s_\lambda(\gamma)$ the corresponding element in S_Γ by the identification of S_Γ and Λ_Γ. For $\lambda \in \mathcal{P}(\Gamma^*)$, we denote

$$(5.1) \qquad s_\lambda = \prod_{\gamma \in \Gamma^*} s_{\lambda(\gamma)}(\gamma) \in S_\Gamma.$$

Then s_λ is the image under the isometry ch of the character of an irreducible representation χ^λ of Γ_n (cf. [M2]).

Denote by $c_n (c \in \Gamma_*)$ the conjugacy class in Γ_n of elements $(x, s) \in \Gamma_n$ such that s is an n-cycle and the cycle product of x is c. Denote by $\sigma_n(c)$ the class function on Γ_n which takes value $n\zeta_c$ (i.e. the order of the centralizer of an element in the class c_n) on elements in the class c_n and 0 elsewhere. For $\rho = \{m_r(c)\}_{r \geq 1, c \in \Gamma_*} \in \mathcal{P}_n(\Gamma_*)$, $\sigma_\rho = \prod_{r \geq 1, c \in \Gamma_*} \sigma_r(c)^{m_r(c)}$ is the class function on Γ_n which takes value Z_ρ on the conjugacy class of type ρ and 0 elsewhere. Given $\gamma \in R(\Gamma)$, we

denote by $\sigma_n(\gamma)$ the class function on Γ_n which takes value $n\gamma(c)$ on elements in the class $c_n, c \in \Gamma_*$, and 0 elsewhere.

The following lemma is easy to check.

Lemma 5.1. *The map* ch *sends* σ_ρ *to* $a'_{-\rho}$. *In particular, it sends* $\sigma_n(c)$ *to* $a_{-n}(c)$ *in* S_Γ *while sending* $\sigma_n(\gamma)$ *to* $a_{-n}(\gamma)$.

5.2. **The characters** $\eta_n(\gamma)$ **and** $\varepsilon_n(\gamma)$. As we have remarked, the map from $\gamma \in \Gamma^*$ to $\eta_n(\gamma)$ can be extended to be a map ¿from $R(\Gamma)$ to $R(\Gamma_n)$. We will need the following proposition which appeared in [W] in a more general setting. Eqn. (5.2) for $\gamma \in \Gamma^*$ is well known (cf. [M2]). We present a complete proof here as we will need some formulas appearing in the proof later on.

Proposition 5.1. *For any* $\gamma \in R(\Gamma)$, *we have*

$$(5.2) \qquad \sum_{n \geq 0} ch(\eta_n(\gamma))z^n \;=\; \exp\left(\sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma)z^n\right),$$

$$(5.3) \qquad \sum_{n \geq 0} ch(\varepsilon_n(\gamma))z^n \;=\; \exp\left(\sum_{n \geq 1} (-1)^{n-1}\frac{1}{n} a_{-n}(\gamma)z^n\right).$$

Proof. First assume that $\gamma \in R(\Gamma)$ is the character of a representation V_γ of Γ. We calculate the character value of $\eta_n(\gamma)$ at $c_n(c \in \Gamma_*)$ first. Take $(g, s) \in \Gamma_n$, where $g = (g_1, \ldots, g_n) \in \Gamma^n$ and s is an n-cycle, say $s = (12 \ldots n)$. Denote by $e_1, \ldots, e_k$ a basis of V_γ, and we write $ge_j = \sum_i a_{ij}(g)e_i$, $a_{ij}(g) \in \mathbb{C}$. It follows that

$$(g, s)(e_{j_1} \otimes \ldots \otimes e_{j_n}) = g_1(e_{j_n}) \otimes g_2(e_{j_1}) \ldots \otimes g_n(e_{j_{n-1}}),$$

in which the coefficient of $e_{j_1} \otimes \ldots \otimes e_{j_n}$ is

$$a_{j_1 j_n}(g_1) a_{j_2 j_1}(g_2) \ldots a_{j_n j_{n-1}}(g_n).$$

Thus we obtain

$$\begin{aligned} \eta_n(\gamma)(c_n) &= \text{trace } a(g_n)a(g_{n-1}) \ldots a(g_1) \\ &= \text{trace } a(g_n g_{n-1} \ldots g_1) = \gamma(c) \end{aligned}$$

since (g, s) is in the conjugacy class c_n which means $g_n g_{n-1} \ldots g_1$ lies in $c \in \Gamma_*$. Since the sign character of Γ_n takes value $(-1)^{n-1}$ at c_n, we have

$$\varepsilon_n(\gamma)(c_n) = (-1)^{n-1}\gamma(c).$$

Given $x \times y \in \Gamma_n$, where $x \in \Gamma_r$ and $y \in \Gamma_{n-r}$, we clearly have $\eta_n(\gamma)(x \times y) = \eta_n(\gamma)(x)\eta_n(\gamma)(y)$. Thus it follows that if $x \in \Gamma_n$ is of

type ρ, then

(5.4)
$$\eta_n(\gamma)(x) = \prod_{c\in\Gamma_*} \gamma(c)^{l(\rho(c))},$$

(5.5)
$$\varepsilon_n(\gamma)(x) = (-1)^n \prod_{c\in\Gamma_*} (-\gamma(c))^{l(\rho(c))},$$

where $||\rho|| = n$. Putting (5.4) into a generating function, we have

$$
\begin{aligned}
\sum_{n\geq 0} \mathrm{ch}(\eta_n(\gamma))z^n &= \sum_{\rho} Z_\rho^{-1} \prod_{c\in\Gamma_*} \gamma(c)^{l(\rho(c))} a'_{-\rho(c)} z^{||\rho||} \\
&= \prod_{c\in\Gamma_*} \left(\sum_\lambda (\zeta_c^{-1}\gamma(c))^{l(\lambda)} z_\lambda^{-1} a_{-\lambda}(c) z^{|\lambda|} \right) \\
&= \exp\left(\sum_{n\geq 1} \frac{1}{n} \sum_{c\in\Gamma_*} \zeta_c^{-1}\gamma(c)a_{-n}(c)z^n \right) \\
&= \exp\left(\sum_{n\geq 1} \frac{1}{n} a_{-n}(\gamma)z^n \right).
\end{aligned}
$$

In a similar manner we can prove (5.3) by using (5.5).

Note that (5.3) can be obtained from (5.2) by substituting γ with $-\gamma$ and z with $-z$. Let β, γ be the characters of two representations of Γ. It follows from (3.4) that

$$
\begin{aligned}
&\sum_{n\geq 0} \mathrm{ch}(\eta_n(\beta-\gamma))z^n \\
&= \left(\sum_{n\geq 0} \mathrm{ch}(\eta_n(\beta))z^n \right) \cdot \left(\sum_{n\geq 0} -\mathrm{ch}(\varepsilon_n(\gamma))(-z)^n \right) \\
&= \exp\left(\sum_{n\geq 1} \frac{1}{n} a_{-n}(\beta)z^n \right) \cdot \exp\left(\sum_{n\geq 1} \frac{1}{n} a_{-n}(-\gamma)z^n \right) \\
&= \exp\left(\sum_{n\geq 1} \frac{1}{n} a_{-n}(\beta-\gamma)z^n \right).
\end{aligned}
$$

Therefore the proposition holds for $\beta - \gamma$, and so for any element $\gamma \in R(\Gamma)$. $\qquad\square$

Remark 5.2. Formulas (5.2-5.3) are equivalent, since (5.3) can be obtained (5.3) from by substituting γ by $-\gamma$ and z by $-z$.

Corollary 5.3. *The formula (5.4) holds for any $\gamma \in R(\Gamma)$. In particular $\eta_n(\xi)$ is self-dual.*

Componentwise, we obtain

$$
\begin{aligned}
\mathrm{ch}(\eta_n(\gamma)) &= \sum_\lambda \frac{1}{z_\lambda} a_{-\lambda}(\gamma), \\
\mathrm{ch}(\varepsilon_n(\gamma)) &= \sum_\lambda \frac{1}{z_\lambda} (-1)^{|\lambda|-l(\lambda)} a_{-\lambda}(\gamma),
\end{aligned}
$$

where the sum runs over all the partitions λ of n.

5.3. **Isometry between R_Γ and S_Γ.** It is well known that there exists a natural Hopf algebra structure on the symmetric algebra S_Γ with the usual multiplication and a comultiplication Δ characterized by

$$
\Delta(a_n(\gamma)) = a_n(\gamma) \otimes 1 + 1 \otimes a_n(\gamma).
$$

Recall that we have defined a Hopf algebra structure on R_Γ in Sect. 2. The following proposition is easy to check.

Proposition 5.2. *The characteristic map $ch : R_\Gamma \longrightarrow S_\Gamma$ is an isomorphism of Hopf algebras.*

Recall that we have defined a bilinear form $\langle \ , \ \rangle_\xi$ on R_Γ and a bilinear form on S_Γ denoted by $\langle \ , \ \rangle'_\xi$. The lemma below follows from our definition of $\langle \ , \ \rangle'_\xi$ and the comultiplication Δ.

Lemma 5.4. *The bilinear form $\langle \ , \ \rangle'_\xi$ on S_Γ can be characterized by the following two properties:*
1). $\langle a_{-n}(\beta), a_{-m}(\gamma) \rangle'_\xi = \delta_{n,m} \langle \beta, \gamma \rangle'_\xi, \quad \beta, \gamma \in \Gamma^*$.
2). $\langle fg, h \rangle'_\xi = \langle f \otimes g, \Delta h \rangle'_\xi$, where $f, g, h \in S_\Gamma$, and the bilinear form on the r.h.s of 2), which is defined on $S_\Gamma \otimes S_\Gamma$, is induced from $\langle \ , \ \rangle'_\xi$ on S_Γ.

Theorem 5.5. *The characteristic map ch is an isometry from the space $(R_\Gamma, \langle \ , \ \rangle_\xi)$ to $(S_\Gamma, \langle \ , \ \rangle'_\xi)$.*

Proof. By Corollary 5.3, the character value of $\eta_n(\xi)$ at an element x of type ρ is

$$
\eta_n(\xi)(x) = \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}.
$$

Thus it follows by definition of the weighted bilinear form

$$
\begin{aligned}
\langle \sigma_{\rho'}, \sigma_{\overline{\rho}} \rangle_\xi &= \sum_{\mu \in \mathcal{P}_n(\Gamma_*)} Z_\mu^{-1} \xi(c_\mu) \sigma_{\rho'}(c_\mu) \sigma_{\overline{\rho}}(c_\mu^{-1}) \\
&= Z_\rho^{-1} \xi(c_\rho) \delta_{\mu,\rho'} Z_{\rho'} \delta_{\mu,\rho} Z_\rho \\
&= \delta_{\rho,\rho'} Z_\rho^{-1} \xi(c_\rho) Z_\rho Z_\rho \\
&= \delta_{\rho,\rho'} Z_\rho \prod_{c \in \Gamma_*} \xi(c)^{l(\rho(c))}.
\end{aligned}
$$

Here c_μ denotes the conjugacy class in Γ_n of type μ. By Lemma 5.1 and the formula (4.3), we see that

$$
\langle \sigma_{\rho'}, \sigma_{\overline{\rho}} \rangle_\xi = \langle a_{-\rho'}, a_{-\overline{\rho}} \rangle'_\xi = \langle \mathrm{ch}(\sigma_{\rho'}), \mathrm{ch}(\sigma_{\overline{\rho}}) \rangle'_\xi.
$$

Since $\sigma_\rho, \rho \in \mathcal{P}(\Gamma_*)$ consist a $\mathbb{C}$-basis of R_Γ, we have established that $\mathrm{ch} : R_\Gamma \longrightarrow S_\Gamma$ is an isometry. $\qquad \square$

Thanks to the isometry established above, we will write $\langle \ , \ \rangle_\xi$ for $\langle \ , \ \rangle'_\xi$ on S_Γ from now on.

Remark 5.6. In the special case when ξ is trivial, Theorem 5.5 was established in [M1, M2] once we identify S_Γ with the space of symmetric functions parameterized by Γ^*.

Remark 5.7. The standard Hermitian form on $R(\Gamma_n)$ and therefore on R_Γ is compatible via the characteristic map ch with the hermitian form characterized by (4.2) on S_Γ.

6. Vertex operators and R_Γ

6.1. **A 2-cocycle on an integral lattice.** Let Q be an integral lattice with a symmetric bilinear form $\langle \ , \ \rangle$. One can easily check that the map from Q to $\mathbb{Z}/2\mathbb{Z}$ given by $\alpha \mapsto (-1)^{\langle \alpha, \alpha \rangle}$ is a group homomorphism. This homomorphism naturally induces a $\mathbb{Z}/2\mathbb{Z}$-gradation on Q.

Let $\epsilon : Q \times Q \longrightarrow \mathbb{C}^\times$ be such that

$$
\begin{aligned}
(6.1) \qquad \epsilon(\gamma, 0) &= \epsilon(0, \gamma) = 1, \\
(6.2) \qquad \epsilon(\alpha, \beta) \epsilon(\alpha + \beta, \gamma) &= \epsilon(\alpha, \beta + \gamma) \epsilon(\beta, \gamma).
\end{aligned}
$$

i.e. ϵ is a 2-cocycle of the group Q with values in $\mathbb{C}^\times$. Another 2-cocycle ϵ' is equivalent to ϵ if and only if there exists $\epsilon_\alpha (\alpha \in R_{\mathbb{Z}}(\Gamma))$ such that

$$
\epsilon'(\alpha, \beta) = \epsilon_\alpha \epsilon_\beta \epsilon_{\alpha+\beta}^{-1} \epsilon(\alpha, \beta).
$$

The group of equivalent classes of 2-cocycles is the second cohomology group $H^2(Q, \mathbb{C}^\times)$.

We introduce

$$B_\epsilon(\alpha, \beta) = \epsilon(\alpha, \beta)\epsilon(\beta, \alpha)^{-1}.$$

Clearly B_ϵ is skew-symmetric: $B_\epsilon(\alpha, \beta) = B_\epsilon(\beta, \alpha)^{-1}$. It follows from Eqns. (6.1) and (6.2) that B_ϵ is bimultiplicative:

$$\begin{aligned} B_\epsilon(\alpha + \beta, \gamma) &= B_\epsilon(\alpha, \gamma)B_\epsilon(\beta, \gamma), \\ B_\epsilon(\alpha, \beta + \gamma) &= B_\epsilon(\alpha, \beta)B_\epsilon(\alpha, \gamma). \end{aligned}$$

The map $\epsilon \mapsto B_\epsilon$ gives rise to a homomorphism from $H^2(Q, \mathbb{C}^\times)$ to the group of bimultiplicative skew-symmetric functions on $Q \times Q$. It is easy to show that the homomorphism h is indeed an isomorphism.

Proposition 6.1. [FK] *There exists a unique 2-cocyle $\epsilon : Q \times Q \longrightarrow \mathbb{C}^\times$ up to equivalence such that*

$$B_\epsilon(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}.$$

6.2. **Vertex Operators** $X(\gamma, z)$. Endowed with the weighted bilinear form $\langle \ , \ \rangle_\xi$, the lattice $R_\mathbb{Z}(\Gamma)$ is an integral lattice. We will always associate a 2-cocycle ϵ as in Proposition 6.1 to the integral lattice $(R_\mathbb{Z}(\Gamma), \langle \ , \ \rangle_\xi)$ (and its sublattices). Denote by $\mathbb{C}[R_\mathbb{Z}(\Gamma)]$ the group algebra generated by e^γ, $\gamma \in R_\mathbb{Z}(\Gamma)$. We introduce two special operators acting on $\mathbb{C}[R_\mathbb{Z}(\Gamma)]$, namely a ($\epsilon$-twisted) multiplication operator e^α:

$$e^\alpha . e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}, \quad \alpha, \beta \in R_\mathbb{Z}(\Gamma),$$

and a differentiation operator ∂_γ:

$$\partial_\gamma e^\beta = \langle \gamma, \beta \rangle_\xi e^\beta, \quad \alpha, \beta \in R_\mathbb{Z}(\Gamma).$$

We extend these two operators acting on $\mathbb{C}[R_\mathbb{Z}(\Gamma)]$ to the following space

$$\mathcal{F}_\Gamma = R_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)],$$

by letting them act on the R_Γ part trivially.

Introduce the operators $H_{\pm n}(\gamma), E_{\pm n}(\gamma), \gamma \in R(\Gamma), n > 0$ as the following compositions of maps:

$$H_{-n}(\gamma) \ : \ R(\Gamma_m) \xrightarrow{\eta_n(\gamma)\otimes} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m})$$

$$E_{-n}(\gamma) \ : \ R(\Gamma_m) \xrightarrow{\varepsilon_n(\gamma)\otimes} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m})$$

$$E_n(\gamma) \ : \ R(\Gamma_m) \xrightarrow{Res} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{\langle \varepsilon_n(\gamma), \cdot \rangle_\xi} R(\Gamma_{m-n})$$

$$H_n(\gamma) \ : \ R(\Gamma_m) \xrightarrow{Res} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{\langle \eta_n(\gamma), \cdot \rangle_\xi} R(\Gamma_{m-n}).$$

Define

$$H_+(\gamma, z) = \sum_{n>0} H_{-n}(\gamma) z^n, \qquad E_+(\gamma, z) = \sum_{n>0} E_{-n}(\gamma)(-z)^n,$$

$$E_-(\gamma, z) = \sum_{n>0} E_n(\gamma)(-z)^{-n}, \qquad H_-(\gamma, z) = \sum_{n>0} H_n(\gamma) z^{-n}.$$

Further define operators $X_n(\gamma), n \in \mathbb{Z} + \langle \gamma, \gamma \rangle_\xi / 2$ by the following generating functions:

$$\begin{aligned}
(6.3) \qquad X^+(\gamma, z) &\equiv X(\gamma, z) \\
&= \sum_{n \in \mathbb{Z} + \langle \gamma, \gamma \rangle_\xi / 2} X_n(\gamma) z^{-n - \langle \gamma, \gamma \rangle_\xi / 2} \\
&= H_+(\gamma, z) E_-(\gamma, z) e^\gamma z^{\partial_\gamma}.
\end{aligned}$$

Sometimes we denote

$$\begin{aligned}
X^-(\gamma, z) &\equiv X(-\gamma, z) \\
&= \sum_{n \in \mathbb{Z} + \langle \gamma, \gamma \rangle_\xi / 2} X_n^-(\gamma) z^{-n - \langle \gamma, \gamma \rangle_\xi / 2}.
\end{aligned}$$

One easily sees that the operators $X_n(\gamma)$ are well-defined operators acting on the space $\mathcal{F}_\Gamma$. We extend the bilinear form $\langle \ , \ \rangle_\xi$ on R_Γ to $\mathcal{F}_\Gamma$ by letting

$$\langle a.e^\alpha, b.e^\beta \rangle_\xi = \langle a, b \rangle_\xi \delta_{\alpha, \beta}, \quad a, b \in R_\Gamma, \ \alpha, \beta \in R_\mathbb{Z}(\Gamma).$$

We extend the $\mathbb{Z}_+$-gradation on R_Γ to a $\frac{1}{2}\mathbb{Z}_+$-gradation on $\mathcal{F}_\Gamma$ by letting

$$\deg a_{-n}(\gamma) = n, \quad \deg e^\gamma = \frac{1}{2} \langle \gamma, \gamma \rangle_\xi.$$

Similarly we extend the bilinear form $\langle \ , \ \rangle_\xi$ to the space

$$V_\Gamma = S_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)]$$

and extend the $\mathbb{Z}_+$-gradation on S_Γ to a $\frac{1}{2}\mathbb{Z}_+$-gradation on V_Γ.

We extend the characteristic map ch to an isometry ¿from $\mathcal{F}_\Gamma$ to V_Γ by the identity operator on the factor $\mathbb{C}[R_\mathbb{Z}(\Gamma)]$, which is denoted again by ch.

6.3. **Heisenberg algebra and R_Γ.** We define $\widetilde{a}_{-n}(\gamma), n > 0$ to be a map ¿from R_Γ to itself by the following composition

$$R(\Gamma_m) \xrightarrow{\sigma_n(\gamma) \otimes} R(\Gamma_n) \bigotimes R(\Gamma_m) \xrightarrow{Ind} R(\Gamma_{n+m}).$$

We also define $\widetilde{a}_n(\gamma), n > 0$ to be a map from R_Γ to itself as the composition

$$R(\Gamma_m) \xrightarrow{Res} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \xrightarrow{\langle \sigma_n(\gamma), \cdot \rangle_\xi} R(\Gamma_{m-n}).$$

We denote by R_Γ^0 the radical of the bilinear form $\langle \, , \, \rangle_\xi$ in R_Γ and denote by $\overline{R}_\Gamma$ the quotient R_Γ/R_Γ^0. $\overline{R}_\Gamma$ inherits the bilinear form $\langle \, , \, \rangle_\xi$ from R_Γ.

Theorem 6.1. *R_Γ is a representation of the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ by letting $a_n(\gamma)$ $(n \in \mathbb{Z}\backslash 0)$ act as $\widetilde{a}_n(\gamma)$, $a_0(\gamma)$ as 0 and C as 1. R_Γ^0 is a subrepresentation of R_Γ over $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ and the quotient $\overline{R}_\Gamma$ is irreducible. The characteristic map ch is an isomorphism of R_Γ (resp. R_Γ^0, $\overline{R}_\Gamma$) and S_Γ (resp. S_Γ^0, $\overline{S}_\Gamma$) as representations over $\widehat{\mathfrak{h}}_{\Gamma,\xi}$.*

Remark 6.2. For a Γ-space X, the wreath product Γ_n acts on the n-th direct product X^n. A Heisenberg algebra was defined in [W] to act on a direct sum of equivariant K-theory $\bigoplus_{n\geq 0} K_{\Gamma_n}(X^n) \bigotimes \mathbb{C}$. In view of Corollary 5.3, our Heisenberg algebra here is a special case of the Heisenberg algebra constructed in [W] when X is a point, cf. [W] for a proof.

6.4. **Vertex operators and Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$.** The relation between the vertex operators defined in (6.3) and the Heisenberg algebra $\widehat{\mathfrak{h}}_{\Gamma,\xi}$ is revealed in the following theorem.

Theorem 6.3. *For any $\gamma \in R(\Gamma)$, we have*

$$ch\big(H_+(\gamma, z)\big) = \exp\bigg(\sum_{n\geq 1} \frac{1}{n} a_{-n}(\gamma) z^n\bigg),$$

$$ch\big(E_+(\gamma, z)\big) = \exp\bigg(-\sum_{n\geq 1} \frac{1}{n} a_{-n}(\gamma) z^n\bigg),$$

$$ch\big(H_-(\gamma, z)\big) = \exp\bigg(\sum_{n\geq 1} \frac{1}{n} a_n(\gamma) z^{-n}\bigg),$$

$$ch\big(E_-(\gamma, z)\big) = \exp\bigg(-\sum_{n\geq 1} \frac{1}{n} a_n(\gamma) z^{-n}\bigg).$$

Proof. The first and second identities were essentially established in Proposition 5.1 together with Lemma 5.1. The only minor difference is that the components appearing in these two identities are regarded as operators acting on R_Γ and S_Γ.

Note that by definition the adjoints of $E_+(\gamma, z)$ and $H_+(\gamma, z)$ with respect to the bilinear form $\langle\ ,\ \rangle_\xi$ are $E_-(\gamma, z^{-1})$ and $H_-(\gamma, z^{-1})$, respectively. The third and fourth identities are obtained by applying the adjoint functor to the first two identities. $\square$

Putting all pieces together, we have

$$
\begin{aligned}
&\mathrm{ch}\big(X(\gamma, z)\big) \\
=\ &\exp\Big(\sum_{n\geq 1} \frac{1}{n} a_{-n}(\gamma)z^n\Big)\ \exp\Big(-\sum_{n\geq 1}\frac{1}{n} a_n(\gamma)z^{-n}\Big) e^\gamma z^{\partial_\gamma}.
\end{aligned}
$$

7. Vertex representations and the McKay correspondence

7.1. Product of two vertex operators. We first note that

$$
X_n(\gamma)^* = X_{-n}(-\gamma), \quad n \in \mathbb{Z} + \langle\gamma, \gamma\rangle_\xi/2.
$$

The normal ordered product $:X(\alpha, z)X(\beta, w):$, $\alpha, \beta \in R(\Gamma)$ of two vertex operators is defined as follows:

$$
\begin{aligned}
&:X(\alpha, z)X(\beta, w): \\
=\ &\exp\Big(\sum_{n\geq 1}\frac{1}{n}\big(a_{-n}(\alpha)z^n + a_{-n}(\beta)w^n\big)\Big)\ \cdot \\
&\cdot \exp\Big(-\sum_{n\geq 1}\frac{1}{n}\big(a_n(\gamma)z^{-n} + a_n(\beta)w^{-n}\big)\Big) e^{\alpha+\beta} z^{\partial_\alpha} w^{\partial_\beta}.
\end{aligned}
$$

The term $(z-w)^{\langle\alpha,\beta\rangle_\xi}$ in the following theorem is understood as the formal series expansion (cf. [FLM])

$$
z^{\langle\alpha,\beta\rangle_\xi} \sum_{k\geq 0}\binom{\langle\alpha,\beta\rangle_\xi}{k}(-z^{-1}w)^k.
$$

This applies to similar expressions in similar contexts below.

Theorem 7.1. *For $\alpha, \beta \in R_\mathbb{Z}(\Gamma)$, we have the following identity for a product of two vertex operators:*

$$
X(\alpha, z)X(\beta, w)\ =\ \epsilon(\alpha, \beta):X(\alpha, z)X(\beta, w):(z-w)^{\langle\alpha,\beta\rangle_\xi}.
$$

Proof. We first compute

$$E_-(\alpha, z)H_+(\beta, w)$$

$$= \exp\left(-\sum_{n\geq 1}\frac{1}{n}a_n(\alpha)z^{-n}\right)\exp\left(\sum_{n\geq 1}\frac{1}{n}a_{-n}(\beta)w^n\right)$$

$$= \exp\left(\sum_{n\geq 1}\frac{1}{n}a_{-n}(\beta)w^n\right)\exp\left(-\sum_{n\geq 1}\frac{1}{n}a_n(\alpha)z^{-n}\right)\cdot$$

$$\cdot \exp\left(-\langle\alpha, \beta\rangle_\xi\sum_{n\geq 1}\frac{1}{n}z^{-n}w^n\right)$$

$$= \exp\left(\sum_{n\geq 1}\frac{1}{n}a_{-n}(\beta)w^n\right)\exp\left(-\sum_{n\geq 1}\frac{1}{n}a_n(\alpha)z^{-n}\right)(1 - z^{-1}w)^{\langle\alpha,\beta\rangle_\xi}.$$

The second identity above uses the formula

$$\exp(A)\exp(B) = \exp(B)\exp(A)\exp([A, B])$$

since $[A, B]$ here commutes with A and B.

On the other hand, we have

$$z^{\partial_{\gamma_i}}e^{\gamma_j} = z^{a_{ij}}e^{\gamma_j}z^{\partial_{\gamma_i}}.$$

Combining with Eqns. (6.1) and (6.2) we compute

$$X(\alpha, z)X(\beta, w)$$

$$= H_+(\alpha, z)E_-(\alpha, z)e^\alpha z^{\partial_{\alpha_i}}H_+(\beta, w)E_-(\beta, w)e^\beta w^{\partial_\beta}$$

$$= \epsilon(\alpha, \beta)H_+(\alpha, z)H_+(\beta, w)E_-(\alpha, z)E_-(\beta, w)\cdot$$

$$\cdot e^{\alpha+\beta}z^{\langle\alpha,\beta\rangle_\xi}z^{\partial_\alpha}w^{\partial_\beta}\left(1 - \frac{w}{z}\right)^{\langle\alpha,\beta\rangle_\xi}$$

$$= \epsilon(\alpha, \beta) : X(\alpha, z)X(\beta, w) : (z - w)^{\langle\alpha,\beta\rangle_\xi}.$$

$\square$

The following proposition is easy to check.

Proposition 7.1. *Given $\alpha \in R(\Gamma), \beta \in R_\mathbb{Z}(\Gamma)$, we have*

$$[a_n(\alpha), X(\beta, z)] = \langle\alpha, \beta\rangle_\xi X(\beta, z)z^n.$$

7.2. **Affine Lie algebra $\widehat{\mathfrak{g}}$ and toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}$.** Let $\mathfrak{g}$ be a complex simple Lie algebra of ADE type of rank r, with a root system $\overline{\Delta}$ and a set of simple roots $\alpha_1. \ldots, \alpha_r$. Denote by θ the highest root. Denote by $e_\alpha, f_\alpha, h_\alpha, \alpha \in \overline{\Delta}$ the Chevalley generators. Sometimes we will write e_i, f_i and h_i for $e_{\alpha_i}, f_{\alpha_i}$ and h_{α_i} respectively. Denote by $(,)$ the invariant bilinear form on $\mathfrak{g}$ normalized by letting $(\theta, \theta) = 2$. Let $\mathbb{C}[t, t^{-1}]$ be the space of Laurent polynomials in an indeterminate t.

The affine algebra $\widehat{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g} \bigotimes \mathbb{C}[t, t^{-1}]$ by a one-dimensional center with a generator C. As a vector space, $\widehat{\mathfrak{g}} = \mathfrak{g} \bigotimes \mathbb{C}[t, t^{-1}] \bigoplus \mathbb{C}C$. We denote by

$$a(n) = a \otimes t^n, \quad a \in \mathfrak{g}, n \in \mathbb{Z}.$$

The commutation relations in $\widehat{\mathfrak{g}}$ are given as follows:

$$\begin{aligned}
[a(n), b(m)] &= [a, b](n + m) + n\delta_{n,-m}(a, b)C, \\
[C, a(n)] &= 0, \quad a, b \in \mathfrak{g}, n, m \in \mathbb{Z}.
\end{aligned}$$

We denote by $A = (a_{ij})_{0 \le i,j \le r}$ the affine Cartan matrix associated to $\widehat{\mathfrak{g}}$. The matrix $(a_{ij})_{1 \le i,j \le r}$ obtained by deleting the first row and column of A is the Cartan matrix of $\mathfrak{g}$. We fix a 2-cocycle ε on the affine root lattice given in Proposition 6.1.

The so-called *basic* representation V of $\widehat{\mathfrak{g}}$ is an irreducible highest weight representation generated by a highest weight vector which is annihilated by $a(n), n \ge 0, a \in \mathfrak{g}$; C acts on V as the identity operator.

We denote by $\widehat{\widehat{\mathfrak{g}}}$ the toroidal Lie algebra over $\mathbb{C}$ (associated to $\mathfrak{g}$) with the following presentation [MRY]: generators are

$$C, h_i(n), x_n(\pm\alpha_i), n \in \mathbb{Z}, i = 0, \dots, r;$$

relations are given by: C is central, and

$$\begin{aligned}
[h_i(n), h_j(m)] &= na_{ij}\delta_{n,-m}C, \\
[h_i(n), x_m(\pm\alpha_j)] &= \pm a_{ij}x_{n+m}(\pm\alpha_j), \\
(7.1) \qquad [x_n(\alpha_i), x_m(-\alpha_j)] &= \delta_{ij}\epsilon(\alpha_i, -\alpha_i)\{h_i(n + m) + n\delta_{n,-m}C\}, \\
[x_n(\pm\alpha_i), x_m(\pm\alpha_i)] &= 0, \\
(ad\, x_0(\pm\alpha_i))^{1-a_{ij}}x_m(\pm\alpha_j) &= 0, \quad (i \ne j),
\end{aligned}$$

where $n, m \in \mathbb{Z}$, $i, j = 0, 1, \dots, r$.

There exists a surjective homomorphism from the toroidal algebra $\widehat{\widehat{\mathfrak{g}}}$ to the double loop algebra $\mathfrak{g} \bigotimes \mathbb{C}[s, s^{-1}, t, t^{-1}]$ given by:

$$\begin{aligned}
C &\mapsto 0, & h_i(k) &\mapsto h_i \otimes s^k, & i &= 0, \dots, r; \\
x_n(\alpha_i) &\mapsto e_i \otimes s^n, & x_n(-\alpha_i) &\mapsto f_i \otimes s^n, & i &= 1, \dots, r; \\
x_n(\alpha_0) &\mapsto f_\theta \otimes s^n t, & x_n(-\alpha_0) &\mapsto e_\theta \otimes s^n t^{-1}, & n &\in \mathbb{Z}.
\end{aligned}$$

7.3. A new form of McKay correspondence.
In this subsection we set Γ to be a finite subgroup of SU_2 and ξ to be the virtual character $2\gamma_0 - \pi$ of Γ, where π is the character of the two-dimensional natural representation in which Γ embeds in SU_2. We recall that the matrix $A = (a_{ij})_{0 \le i,j \le r}$ in Sect. 3.3 is the Cartan matrix for the corresponding affine Lie algebra $\widehat{\mathfrak{g}}$.

The following theorem provides a finite group realization of the vertex representation of the toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}$ on $\mathcal{F}_\Gamma \cong V_\Gamma$.

Theorem 7.2. *A vertex representation of the toroidal Lie algebra $\widehat{\widehat{\mathfrak{g}}}$ is defined on the space $\mathcal{F}_\Gamma$ by letting*

$$x_n(\alpha_i) \mapsto X_n(\gamma_i), \qquad x_n(-\alpha_i) \mapsto X_n(-\gamma_i),$$
$$h_i(n) \mapsto a_n(\gamma_i), \qquad C \mapsto 1,$$

where $n \in \mathbb{Z}, 0 \le i \le r$.

Proof. The identity for the product of two vertex operators in Theorem 7.1 implies the commutation relations between the components of vertex operators either by formal calculus or the Cauchy residue formula (see [FLM]). Taking into account Proposition 7.1 one deduces that the correspondence defined in Theorem 7.2 indeed realizes a representation of the toroidal algebra $\widehat{\widehat{\mathfrak{g}}}$, (cf. [F2, MRY]). $\square$

Recall that $\delta = \sum_{i=0}^r d_i \gamma_i$ generates the one-dimensional radical $R_\mathbb{Z}^0$ of the bilinear form $\langle \ , \ \rangle_\xi$ in $R_\mathbb{Z}(\Gamma)$. The lattice $R_\mathbb{Z}(\Gamma)$ in this case can be identified with the root lattice for the corresponding affine Lie algebra. The quotient lattice $R_\mathbb{Z}(\Gamma)/R_\mathbb{Z}^0$ inherits a positive definite integral bilinear form. Denote by $\overline{\Gamma}^*$ the set of non-trivial irreducible characters of Γ:

$$\overline{\Gamma}^* = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}.$$

Denote by $\overline{R}_\mathbb{Z}(\Gamma)$ the sublattice in $R_\mathbb{Z}(\Gamma)$ generated by $\overline{\Gamma}^*$. The subset of elements in $\overline{R}_\mathbb{Z}(\Gamma)$ of length $\sqrt{2}$ can be identified with the finite root system $\overline{\Delta}$. Equipped with bilinear form $\langle \ , \ \rangle_\xi$, $\overline{R}_\mathbb{Z}(\Gamma)$ is obviously isomorphic to $R_\mathbb{Z}(\Gamma)/R_\mathbb{Z}^0$, which is in turn identified with the root lattice of Lie algebra $\mathfrak{g}$. Furthermore $R_\mathbb{Z}(\Gamma)$ can be written as a direct sum of two integral lattices $\overline{R}_\mathbb{Z}(\Gamma) \bigoplus \mathbb{Z}\delta$.

Denote by $Sym(\overline{\Gamma}^*)$ the symmetric algebra generated by $a_{-n}(\gamma_i)$, $n > 0$, $i = 1, \ldots, r$. Equipped with the bilinear form $\langle \ , \ \rangle_\xi$, $Sym(\overline{\Gamma}^*)$ is isometric to $\overline{S}_\Gamma$ which is in turn isometric to $\overline{R}_\Gamma$ as well.

We define

$$\overline{V}_\Gamma = \overline{S}_\Gamma \bigotimes \mathbb{C}[R_\mathbb{Z}(\Gamma)/R_\mathbb{Z}^0] \cong Sym(\overline{\Gamma}^*) \bigotimes \mathbb{C}[\overline{R}_\mathbb{Z}(\Gamma)],$$
$$\overline{\mathcal{F}}_\Gamma = \overline{R}_\Gamma \bigotimes \mathbb{C}[\overline{R}_\mathbb{Z}(\Gamma)] \cong \overline{V}_\Gamma.$$

The space V_Γ associated to the lattice $R_\mathbb{Z}(\Gamma)$ is isomorphic to the tensor product of the space $\overline{V}_\Gamma$ associated to $\overline{R}_\mathbb{Z}(\Gamma)$ and the space associated to the rank 1 lattice $\mathbb{Z}\delta$ equipped with the zero bilinear form.

The identity for a product of vertex operators $X(\gamma, z)$ associated to $\gamma \in \overline{\Delta}$ (cf. Theorem 7.2) imply that $\overline{V}_\Gamma$ affords a realization of

the vertex representation of $\widehat{\mathfrak{g}}$ on $\overline{V}_\Gamma$ [FK, S1]. The following theorem provides a direct link from the finite group $\Gamma \in SU_2$ to the affine Lie algebra $\widehat{\mathfrak{g}}$ and thus it can be regarded as a new form of McKay correspondence.

Theorem 7.3. *The operators $X_n(\gamma), \gamma \in \overline{\Delta}, a_n(\gamma_i), i = 1, 2, \ldots, r, n \in \mathbb{Z}$ define an irreducible representation of the affine Lie algebra $\widehat{\mathfrak{g}}$ on $\overline{\mathcal{F}}_\Gamma$ isomorphic to the basic representation. In particular the operators $X_0(\gamma), \gamma \in \overline{\Delta}, a_0(\gamma_i), i = 1, 2, \ldots, r$ define a representation of the simple Lie algebra $\mathfrak{g}$. $\overline{V}_\Gamma$ carries a distinguished basis $\chi^\rho \otimes e^\gamma, \rho \in \mathcal{P}(\overline{\Gamma}^*), \gamma \in \overline{R}_\mathbb{Z}(\Gamma)$.*

8. VERTEX OPERATORS AND IRREDUCIBLE CHARACTERS OF Γ_n

In this section we specialize ξ to be the trivial character γ_0 of Γ.

8.1. **Algebra of vertex operators for $\xi = \gamma_0$.** In this case the weighted bilinear form reduces to the standard one $\langle\ ,\ \rangle$ and $R_\mathbb{Z}(\Gamma)$ is isomorphic to the lattice $\mathbb{Z}^n$ with the standard integral bilinear form. Recall that $\langle\gamma_i, \gamma_j\rangle = \delta_{ij}$. The 2-cocycle ϵ can be chosen by letting $\epsilon(\gamma_i, \gamma_j) = 1$ if $i \leq j$ and -1 if $i > j$. It follows by standard argument that the identity for product of vertex operators given in Theorem 7.1 implies the following anti-commutation relations (cf. [F1]). The bracket $\{\ ,\ \}$ below denotes the anti-commutator.

Theorem 8.1. *The operators $X_n^+(\gamma_i), X_n^-(\gamma_i)$ ($n \in \mathbb{Z} + \frac{1}{2}, 0 \leq i \leq r$) generate a Clifford algebra:*

$$\{X_m^+(\gamma_i), X_n^+(\gamma_j)\} = 0,$$
$$\{X_m^-(\gamma_i), X_n^-(\gamma_j)\} = 0,$$
(8.1)
$$\{X_m^+(\gamma_i), X_n^-(\gamma_i)\} = \delta_{ij}\delta_{m,-n}.$$

8.2. **Character tables of Γ_n and vertex operators.** We now construct a special orthonormal basis in V_Γ and then interpret them as irreducible characters of Γ_n. First we have the following simple lemma, cf. [J3] for a proof.

Lemma 8.2. *For $m \in \mathbb{Z} + 1/2$, $\alpha \in R(\Gamma)$ and $\gamma \in \Gamma^*$, we have*

$$X_m(\gamma)e^\alpha = \delta_{m,-\langle\alpha,\gamma\rangle-1/2}e^{\alpha+\gamma}, \quad m \geq -\langle\gamma, \alpha\rangle - 1/2,$$
$$X_m^-(\gamma)e^\alpha = \delta_{m,\langle\alpha,\gamma\rangle-1/2}e^{\alpha-\gamma}, \quad m \geq \langle\gamma, \alpha\rangle - 1/2.$$

Consequently we have

$$X_{-m-\langle\gamma,\alpha\rangle+1/2}(\gamma)X_{-m-\langle\gamma,\alpha\rangle+3/2}(\gamma)\cdots X_{-\langle\gamma,\alpha\rangle-1/2}(\gamma)e^\alpha = e^{\alpha+m\gamma},$$
$$X_{-m+\langle\gamma,\alpha\rangle+1/2}^-(\gamma)X_{-m+\langle\gamma,\alpha\rangle+3/2}^-(\gamma)\cdots X_{\langle\gamma,\alpha\rangle-1/2}^-(\gamma)e^\alpha = e^{\alpha-m\gamma}.$$

For a m-tuple index $\phi = (\phi_1, \cdots, \phi_m) \in (\mathbb{Z} + 1/2)^m$ we denote
$$X_\phi(\gamma) = X_{\phi_1}(\gamma) \cdots X_{\phi_m}(\gamma)$$
We will use similar notations for other vertex operators.

Let $\delta_l = (l-1, l-2, \cdots, 1, 0)$ be the special partition of length l. We often omit the subscript if the meaning is clear from the context. Given $\lambda \in \mathcal{P}(\Gamma^*)$, we define
$$\omega(\lambda) = \sum_{\gamma \in \Gamma^*} l(\lambda(\gamma))\gamma \in R_\mathbb{Z}(\Gamma),$$
and
$$s_{\lambda,\alpha} = \prod_{\gamma \in \Gamma^*} X_{-\lambda(\gamma)-\delta-(\langle\gamma,\alpha\rangle+1/2)\mathbf{1}}(\gamma).e^\alpha,$$
where the order of $\prod_{\gamma \in \Gamma^*}$ is understood as a product over $\gamma_0, \gamma_1, \ldots, \gamma_r$ from the left to the right.

Recall that $s_\lambda \in S_\Gamma$ for $\lambda \in \mathcal{P}(\Gamma^*)$ is defined in Eq. (5.1).

Theorem 8.3. *The vectors $s_{\lambda,\alpha}$ for $\lambda = (\lambda(\gamma))_{\gamma \in \Gamma^*} \in \mathcal{P}(\Gamma^*), \alpha \in R_\mathbb{Z}(\Gamma)$ form an orthonormal basis in the vector space $\mathcal{F}_\Gamma$, where $\mathbf{1} = (1, \cdots, 1)$ and δ are both of length $l = l(\lambda(\gamma))$. The isometry ch maps the basis vector $s_{\lambda,\alpha}$ to $s_\lambda e^{\alpha+\omega(\lambda)} \in V_\Gamma$.*

Proof. The Clifford algebra structure (8.1) implies that the nonzero elements
$$\prod_{\gamma \in \Gamma^*} X_{-n_1}(\gamma) \cdots X_{-n_l}(\gamma)e^\alpha$$
of distinct indices generate a spanning set in the space V_Γ. To see that they correspond to Schur functions we compute in a way similar to Theorem 7.1 that
$$X(\gamma, z_1) \cdots X(\gamma, z_l)e^\alpha$$
$$=: X(\gamma, z_1) \cdots X(\gamma, z_l) : \prod_{i<j}(1 - z_j z_i^{-1})z^{\delta+(1/2+\langle\gamma,\alpha\rangle)\mathbf{1}}e^\alpha.$$

The method in [J1, J2, J3] implies that this is exactly the generating function of Schur functions under the isomorphism ch. Since the vertex operators corresponding to different $\gamma \in \Gamma^*$ commute up to a sign it follows that the isometry ch maps the basis vector $s_{\lambda,\alpha}$ to $s_\lambda e^{\alpha+\omega(\lambda)} \in V_\Gamma$.

The orthonormality follows readily from the Clifford algebra commutation relations in Theorem 8.1. $\square$

It is a routine computation either by symmetric functions or vertex operator calculus (cf. [J1]) that the complete character table of Γ_n is given by matrix coefficients of the vertex operators $X(\gamma, z)$.

Theorem 8.4. *Given partition-valued functions* $\lambda \in \mathcal{P}(\Gamma^*)$ *and* $\mu \in \mathcal{P}(\Gamma_*)$, *the matrix coefficient*

$$\langle s_{\lambda, -\omega(\lambda)}, a'_{-\mu} \rangle$$

is equal to the character value of the irreducible character of χ^{λ} *associated to* λ *at the conjugacy class of type* μ *in* Γ_n.

References

[B]   R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA **83** (1986) 3068–3071.

[F1]   I. B. Frenkel, *Two constructions of affine algebras and boson-fermion correspondence*, J. Funct. Anal. **44** (1981) 259–327.

[F2]   I. B. Frenkel, *Kac-Moody algebras and dual resonance models*, in *Applications of Group Theory in Physics and Mathematical Physics*, eds. M. Flato et al, Lect. in Appl. Math. **21**, 325–353. AMS, Providence, 1985.

[F3]   I. B. Frenkel, *Lectures on infinite dimensional Lie algebras*, Yale Univ, 1986.

[FK]   I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980) 23–66.

[FLM]   I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex operator algebras and the Monster*, Academic Press, New York, 1988.

[Gr]   I. Grojnowski, *Instantons and affine algebras I: the Hilbert scheme and vertex operators*, Math. Res. Lett. **3** (1996) 275–291.

[J1]   N. Jing, *Vertex operators, symmetric functions and the spin group* Γ_n, J. Alg. **138** (1991) 340–398.

[J2]   N. Jing, *Vertex operators and Hall-Littlewood symmetric functions*, Adv. in Math. **87** (1991) 226–248.

[J3]   N. Jing, *Boson-fermion correspondence for Hall-Littlewood polynomials*, J. Math. Phys. **36** (1995) 7073–7080.

[M1]   I. G. Macdonald, *Polynomial functors and wreath products*, J. Pure Appl. Alg. **18** (1980) 173–204.

[M2]   I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.

[Mc]   J. McKay, *Graphs, singularities and finite groups*, Proc. Sympos. Pure Math. **37**, Amer. Math. Soc, Providence, RI (1980) 183–186.

[MRY]   R. Moody, S. Rao and T. Yokonuma, *Toroidal Lie algebras and vertex representations*, Geom. Dedica. **35** (1990) 283–307.

[N]   H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, 1996.

[S1]   G. Segal, *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. **80** (1981) 301–342.

[S2]   G. Segal, *Equivariant K-theory and symmetric products*, 1996 preprint (unpublished).

[St]   R. Steinberg, *Finite subgroups of* SU_2, *Dynkin diagrams and affine Coxeter elements*, Pacif. J. Math. **118** (1985) 587–598.

[W]   W. Wang, *Equivariant K-theory and wreath products*, MPI preprint # **86**, August 1998; *Equivariant K-theory, wreath products and Heisenberg algebra*, math.QA/9907166, to appear in Duke Math. J.

[Z]   A. Zelevinsky, *Representations of finite classical groups, A Hopf algebra approach*. Lecture Notes in Mathematics, **869**. Springer-Verlag, Berlin-New York, 1981.

FRENKEL: DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06520

JING: DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC 27695-8205
  E-mail address: jing@math.ncsu.edu

WANG: DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CT 06520
  E-mail address: wqwang@math.yale.edu